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LETTER TO THE EDITOR

Trace formulae for the quantum group $GL_q(2)$

B A Kupershmidt

University of Tennessee Space Institute, Tullahoma, TN 37388, USA

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Abstract. On $GL_q(2)$ there exist two types of traces. On $GL_q(n)$, for $n \geqslant 3$, there are none.

Quantum groups are very special non-commutative versions of basic objects of classical mathematics and physics. The adjective 'quantum' reflects connections with the quantization procedure viewed as a deformation with a parameter $q = e^{-\hbar}$, where $\hbar$ can be thought of as the Planck constant in dimensionless units [1]. In the quasiclassical limit, when $q \to 1$ (so that $\hbar \to 0$), the structures of quantum groups turn into multiplicative Poisson structures on Lie groups. An essential and natural part in the study of quantum groups is an analysis of which of the classical structures can be quantized; a negative conclusion is not untypical.

Recently, Ikeda [2] showed that many properties of the classical Toda lattice and its various generalizations can be read off the following formula:

$$\{\operatorname{tr} M^l, \operatorname{tr} M^k\} = 0 \qquad l, k \in \mathbb{N} \tag{1}$$

where $M \in \operatorname{Mat}(n)$ is a $n \times n$ matrix, and the Poisson bracket in (1) is that of the *quasiclassical limit* of the quantum group $GL_q(n)$ [3]:

$$\{M_{i\mu}, M_{j\nu}\} = \text{const}\,[\operatorname{sgn}(i-j) + \operatorname{sgn}(\mu-\nu)] M_{i\nu} M_{j\mu}. \tag{2}$$

Thus, all the Hamiltonians which are traces of various powers of M commute between themselves. the natural question is whether there exists a quantum analogue of this fact. This is the subject of this letter. As we shall see, the answer is negative for $n \geqslant 3$. On the contrary, for the case $n = 2$ the answer is *doubly* positive: there exist precisely *two* definitions of a quantum trace on $GL_q(2)$; for each of these definitions, traces of powers of M commute between themselves.

The plan of the letter is as follows. First we review basic properties of the quantum group $GL_q(2)$ before determining all possible traces on $GL_q(2)$ which guarantee the commutation relation

$$\operatorname{tr}(M)\operatorname{tr}(M^2) = \operatorname{tr}(M^2)\operatorname{tr}(M) \qquad M \in GL_q(2). \tag{3}$$

Next we show that $\operatorname{tr}(M^k)$ is a polynomial in $\operatorname{tr}(M)$ and the quantum determinant $\det_q(M)$, implies that all the traces commute between themselves. Finally we analyse the case of $GL_q(n)$ for $n \geqslant 3$ with the help of various embeddings $GL_q(2) \hookrightarrow GL_q(n)$.

The set $\operatorname{Mat}_q(2)$ of 2×2 q-matrices M,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4}$$

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is defined by the commutation relations [4]:

$$
\begin{array}{lll}
ab = q^{-1} ba & cd = q^{-1} dc & ac = q^{-1} ca \\
bd = q^{\times} \times^{1} db & bc = cb & ad - da = (q^{-1} - q) bc.
\end{array} \tag{5}
$$

The quantum determinant

$$\mathscr{D} = ad - q^{-1} bc = da - qbc \tag{6}$$

commutes with everything:

$$\mathscr{D} u = u \mathscr{D} \qquad u \in \{a, b, c, d\}. \tag{7}$$

One gets $GL_q(2)$ by adjoining $\mathscr{D}^{-1}$ to the associative ring generated by the letters a, b, c, d subject to the relations (5). If

$$M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \tag{8}$$

is another 2×2 q-matrix, and

$$uv = vu \qquad u \in \{a, b, c, d\}, v \in \{a', b', c', d'\} \tag{9}$$

then $M'M$ and MM' are again q-matrices. Also,

$$M^{-1} = \mathscr{D}^{-1} \begin{pmatrix} d & -qb \\ -q^{-1} c & a \end{pmatrix} \in \mathrm{Mat}_{q^{-1}}(2). \tag{10}$$

More generally [5], [6, p 133],

$$M^k \in \mathrm{Mat}_{q^k}(2) \qquad k \in \mathbb{Z}. \tag{11}$$

The case $n = 2$ is very special: a formula similar to (2.7) is not true for $M \in \mathrm{Mat}_q(n)$ $n > 2$.

Trace being a linear form of diagonal elements, we can assume that a quantum version of the classical trace, $\mathrm{tr}(M) = a + d$, has the form

$$\mathrm{tr}_\alpha(M) = a + \alpha d \tag{12}$$

where the (q-dependent) constant $\alpha = \alpha(q)$ is such that $\alpha(1) = 1$. Similarly, we assume that

$$\mathrm{tr}_\beta(M^2) = a_2 + \beta d_2 \tag{13}$$

for some $\beta = \beta(q)$, where

$$M^k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix} \qquad k \in \mathbb{Z} \tag{14}$$

and, in particular,

$$M^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & d^2 + bc \end{pmatrix}. \tag{15}$$

We want to have the equality

$$\mathrm{tr}_\alpha(M)\, \mathrm{tr}_\beta(M^2) = \mathrm{tr}_\beta(M^2)\, \mathrm{tr}_\alpha(M). \tag{16}$$

This is equivalent to the system

$$\alpha(1 + \beta) = (q + q^{-1})\beta \qquad 1 + \beta = \alpha(q + q^{-1}) \tag{17}$$

so that

$$\beta = \alpha^2 \tag{18}$$

$$\alpha = q^{\pm 1}. \tag{19}$$

Thus, on $\mathrm{Mat}_q(2)$ we have two traces:

$$\mathrm{tr}_q(M) = a + qd \tag{20}$$

$$\mathrm{tr}_{q^{-1}}(M) = a + q^{-1}d. \tag{21}$$

Since $M^k \in \mathrm{Mat}_{q^k}(2)$, on $\mathrm{Mat}_{q^k}(2)$ we also have two traces

$$\mathrm{tr}_{\alpha^k}(M^k) = a_k + \alpha^k d_k \qquad \alpha = q^{\pm 1}. \tag{22}$$

This agrees with formula (18).

Remark. Notice that

$$\mathrm{tr}_\alpha(MM') = \mathrm{tr}_\alpha(M'M) \text{ iff } \alpha = 1 \tag{23}$$

so that the quantum trace is no longer symmetric.

Denote

$$t_k = t_k(\alpha) = \mathrm{tr}_{\alpha^k}(M^k) \qquad \alpha = q^{\pm 1} \tag{24}$$

$$t_k^+ = t_k(q) \qquad t_k^- = t_k(q^{-1}). \tag{25}$$

It is easy to verify that

$$t_2 = t_1^2 - 2\alpha\mathcal{D} \tag{26}$$

$$t_3 = t_1^3 - 3\alpha\mathcal{D}t_1. \tag{27}$$

These equalities suggest that

$$t_k \in \mathbb{Z}[t_1, \alpha\mathcal{D}] \qquad k \in \mathbb{N}. \tag{28}$$

Since $M^{-k} = (M^{-1})^k$, and

$$t_{-1} = t_1(\alpha\mathcal{D})^{-1} \tag{29}$$

formulae (10) and (28) imply that

$$t_{-k} \in \mathbb{Z}[t_1, (\alpha\mathcal{D})^{-1}] \qquad k \in \mathbb{N}. \tag{30}$$

Together, formulae (28) and (30) imply, in particular, that

$$t_k t_l = t_l t_k \qquad k, l \in \mathbb{Z} \tag{31}$$

the desired result.

It remains to prove formula (28). This formula, in turn, follows by iteration from the recurrence relation

$$t_{k+1} = t_1 t_k - \alpha\mathcal{D}t_{k-1} \qquad k \in \mathbb{N} \tag{32}$$

and the boundary conditions

$$t_0 = 2 \qquad t_1 = t_1. \tag{33}$$

In particular,

$$t_k = \left(\frac{t_1 + \sqrt{t_1^2 - 4\alpha\mathcal{D}}}{2}\right)^k + \left(\frac{t_1 - \sqrt{t_1^2 - 4\alpha\mathcal{D}}}{2}\right)^k \tag{34}$$

$$= 2^{1-k} \sum_{l=0}^{[k/2]} \binom{k}{2l} t_1^{k-2l}(t_1^2 - 4\alpha\mathcal{D})^l. \tag{35}$$

Remark. In the classical limit $q=1$, formula (32) becomes an identity

$$(\lambda_1^{k+1}+\lambda_2^{k+1})=(\lambda_1+\lambda_2)(\lambda_1^k+\lambda_2^k)-\lambda_1\lambda_2(\lambda_1^{k-1}+\lambda_2^{k-1})$$

for the eigenvalues λ_1 and λ_2 of a 2×2 matrix M.

The identity (32) follows from the characteristic equations for quantum 2-matrices [7]:

$$X^2M^2=\mathrm{tr}(XM)XM-q^{-1}\mathcal{D}\mathbb{1} \qquad (36a)$$

$$M^2Y^2=MY\,\mathrm{tr}(MY)-q\mathcal{D}\mathbb{1} \qquad (36b)$$

$$X=\begin{pmatrix}1 & 0\\ 0 & q^{-1}\end{pmatrix} \qquad Y=\begin{pmatrix}1 & 0\\ 0 & q\end{pmatrix}.$$

Multiplying the equality (36a) from the left by X^{k-1}, from the right by M^{k-1}, and taking the usual trace from the resulting identity, we obtain formula (32) for $\alpha=q^{-1}$. Similarly, multiplying (36b) from the left by M^{k-1}, from the right by Y^{k-1}, and taking traces results in the identity

$$t_{k+1}^{+}=t_k^{+}t_1^{+}-q\mathcal{D}t_{k-1}^{+} \qquad (37)$$

which becomes (32) if we use induction on k and deduce that t_k^+ and t_1^+ commute. Notice also that

$$c^{-1}t_k^{-}c=q^{-k}t_k^{+} \qquad k\in\mathbb{N}. \qquad (38)$$

A $n\times n$ q-matrix N is defined by the property that its every 2×2 submatrix is itself a 2×2 q-matrix satisfying the defining relations (5) for $\mathrm{Mat}_q(2)$. In particular, for any pair of indices $1\leqslant i<j\leqslant n$, letting all the matrix elements of N vanish except

$$N_{ii}=:a \qquad N_{ij}=:b \qquad N_{ji}=:c \qquad N_{jj}=d \qquad (39)$$

we get a legitimate element of $\mathrm{Mat}_q(n)$. Now, for $n\geqslant 3$,

$$N^k\notin \mathrm{Mat}_{q^k}(n) \qquad (40)$$

which hints at the impossibility of defining $\mathrm{tr}_{(\cdot)}(N^k)$.

Suppose $\mathrm{tr}(N)$ and $\mathrm{tr}(N^2)$ are defined as

$$\mathrm{tr}(N)=\sum_i \alpha_i N_{ii} \qquad \mathrm{tr}(N^2)=\sum_i \beta_i (N^2)_{ii} \qquad (41)$$

with some unknown (q-dependent) constants $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n$. Since, at $q=1$, the α_i and β_i all become 1, we can assume that all the α_i and β_i do not vanish. Now, due to the embedding (39), our previous computation applies. Formula (19) then implies that $[\mathrm{tr}(N),\mathrm{tr}(N^2)]=0$ iff

$$\alpha_i/\alpha_j=q^{\pm 1} \qquad i\neq j. \qquad (42)$$

But this is impossible for $q\neq 1$ if the indices i and j can run over three or more different values.

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